Kinetic Equations

Solution to the Exercises

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Exercise 1

Let $f : \mathbb{R}^3 \to \mathbb{R}_+$ be a regular enough function, decaying sufficiently fast at infinity. Prove that the following statements are equivalent:

(i) $\int_{\mathbb{R}^3} Q(f,f) \log f dv = 0$;

(ii) $\log f$ is a collision invariant;

(iii) f is a Maxwellian distribution, i.e. there exist $\rho \in \mathbb{R}$, $\theta > 0$ and $u \in \mathbb{R}^3$ such that $f(v) = \frac{\rho}{(2\pi\theta)^{\frac{3}{2}}} e^{-\frac{|v-u|^2}{2\theta}}$ for all $v \in \mathbb{R}^3$;

(iv) $Q(f,f) = 0$.

Proof. First of all notice that $(ii) \iff (iii)$. From the characterization we gave last time of a collision invariant we have that $\log f(v) = a|v| + b \cdot v + c$ with $a, c \in \mathbb{R}$, $b \in \mathbb{R}^3$. This implies that

$$f(v) = e^{a|v|^2 + b\cdot v + c}. \tag{1}$$

Given that f is decaying at infinity we get $a = -|a|$. Given that we can now write

$$a|v|^2 + b\cdot v + c = -|a||v|^2 + b\cdot v + c = -|a|\left(v - \frac{1}{2|a|}b\right)^2 + \frac{|b|^2}{4|a|} + c. \tag{2}$$

If we define $\theta := \frac{1}{2|a|}$, $u := \frac{1}{2|a|}b$ and $\rho := \left(\frac{\pi}{|a|}\right)^{\frac{3}{2}} e^{\frac{|b|^2}{|a|}+c}$ we can clearly see that f is a Maxwellian distribution. The viceversa comes easily from a similar argument.

We now prove that $(i) \Rightarrow (ii) \Rightarrow (iv) \Rightarrow (i)$ to conclude.

To prove $(i) \Rightarrow (ii)$ recall that we saw in class that we can rewrite $Q(f,f)$ as

$$\begin{aligned}
&\int_{\mathbb{R}^3} Q(f,f)\log f dv = &(3)\\
&= -\frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} \left(f'f'_* - ff_*\right)\log\left(\frac{f'f'_*}{ff_*}\right) B(v-v_*,\omega)\, d\omega\ dv\ dv_* &(4)\\
&= -\frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} ff_*\left(\frac{f'f'_*}{ff_*} - 1\right)\log\left(\frac{f'f'_*}{ff_*}\right) B(v-v_*,\omega)\, d\omega\ dv\ dv_*. &(5)
\end{aligned}$$

Given that $ff_*B(v-v_*,\omega) \geqslant 0$ and that the function $(\lambda - 1)\log(\lambda) \geqslant 0$, $Q(f,f) = 0$ implies that $ff_*\left(\frac{f'f'_*}{ff_*} - 1\right)\log\left(\frac{f'f'_*}{ff_*}\right) B(v-v_*,\omega) = 0$. Given that $ff_*B(v-v_*,\omega) > 0$

for any $\omega \neq 0$ we get that $\frac{f' f'_*}{f f_*} = 1$ almost everywhere. This implies that $\log f$ is collision invariant and therefore (ii).

Using the same expression for $Q(f, f)$, if $\log f$ is collision invariant we have $\frac{f' f'_*}{f f_*} = 1$ and therefore $Q(f, f) = 0$. This proves $(ii) \Rightarrow (iv)$.

Finally $(iv) \Rightarrow (i)$ is trivial, which concludes the proof of the exercise.

$\square$

Exercise 2

Let $v, v_* \in \mathbb{R}^3$, and $\omega \in \mathbb{S}^2$. In the lecture we defined the post-collisional velocities (v', v'_*) associated to the pair of pre-collisional velocities (v, v_*) with the angular parameter ω as:

$$\begin{cases} v' = v - (v - v_*) \cdot \omega \; \omega, \\ v'_* = v_* + (v - v_*) \cdot \omega \; \omega. \end{cases} \tag{6}$$

We denote as $(v', v'_*)(\omega)$ the pair of post-collisional velocities defined by (6). In the literature, one may find another parametrization for the post-collisional velocities, called the *σ-representation*, defined for any $\sigma \in \mathbb{S}^2$ as

$$\begin{cases} v'' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2} \; \sigma, \\ v''_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2} \; \sigma. \end{cases} \tag{7}$$

We denote as $(v'', v''_*)(\sigma)$ the pair of post-collisional velocities defined by (7).

(i) Prove that the two parametrizations are equivalent, i.e. that for any $\omega \in \mathbb{S}^2$, there exists a unique parameter $\sigma \in \mathbb{S}^2$ such that $(v', v'_*)(\omega) = (v'', v''_*)(\sigma)$.

   Prove also that for any $\sigma \in \mathbb{S}^2$ there exists a parameter ω such that $(v', v'_*)(\omega) = (v'', v''_*)(\sigma)$. Is this choice of ω unique? If not, how many possibilities are there for ω for any given σ?

(ii) Represent on a picture, for a given pair of pre-collisional velocities $(v, v_*) \in \mathbb{R}^6$, $v \neq v_*$, and a given angular parameter $\omega \in \mathbb{S}^2$, the associated pair of post-collisional velocities $(v', v'_*)(\omega)$. Represent also the vector σ associated to ω.

(iii) We have seen in the lecture that the collision kernel for the hard sphere model is given by $|(v - v_*) \cdot \omega|$, that is the collision term of the Boltzmann equation writes:

$$Q(f, f) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |(v - v_*) \cdot \omega| \left(f' f'_* - f f_*\right) d\omega \; dv_*. \tag{8}$$

   Prove that in the σ-representation the hard sphere collision kernel is given by $|v - v_*|$, i.e.:

$$Q(f, f) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \frac{|v - v_*|}{2} \left(f'' f''_* - f f_*\right) d\sigma \; dv_*, \tag{9}$$

   (where $f'' = f(v'')$ and $f''_* = f(v''_*)$).

Proof. We first give some general properties of (6) and (7). Suppose first that we have $(v', v'_*)(\omega) = (v', v'_*)(\omega')$ for all $v, v_* \in \mathbb{R}^3$. From the fact that $v'(\omega) = v'(\omega')$ we get immediately that $(v - v_*) \cdot \omega\ \omega = (v - v_*) \cdot \omega'\ \omega'$. This implies first that ω and ω' are colinear, and subsequently that are equal up to a sign. Viceversa, the ω-parametrizations associated to the vectors $\omega, -\omega \in \mathbb{S}^2$ coincide, i.e. $(v', v'_*)(-\omega) = (v', v'_*)(\omega)$. Finally, given that $\omega = \frac{v'_* - v_*}{|v'_* - v_*|}$, ω can always be identified up to a sign, given the transformation.

Suppose now that $(v'', v''_*)(\sigma) = (v'', v''_*)(\sigma')$ for all $v, v_* \in \mathbb{R}^3$. From the fact that $v''(\sigma) - v''_*(\sigma) = v''(\sigma') - v''_*(\sigma')$ we get that $\sigma = \sigma'$. Moreover $\sigma = \frac{v'' - v''_*}{|v - v_*|}$, therefore given the transformation we can always uniquely identify σ.

We get now to point (i). Let $\omega \in \mathbb{S}^2$ be fixed. Then from the properties of above the σ associated to the transformation that sends v, v_* in v', v'_* identifies σ uniquely. On the other hand to each σ we have two corresponding values of ω identified, equal in direction but opposite in sign.

Regarding point (iii), denote as $v'(\omega, v, v_*)$ and $v'_*(\omega, v, v_*)$ the vectors defined through (6), where we made explicit the dependence on v and v_*. First of all we get that

$$v'\left(\omega, v'(\omega, v, v_*), v'_*(\omega, v, v_*)\right) = v'(\omega, v, v_*) - \left(v'(\omega, v, v_*) - v'_*(\omega, v, v_*)\right) \cdot \omega\ \omega \quad (10)$$

$$= v - (v - v_*) \cdot \omega\ \omega - (v - v_* - 2(v - v_*) \cdot \omega\ \omega) \cdot \omega\ \omega = v, \quad (11)$$

$$v'_*\left(\omega, v'(\omega, v, v_*), v'_*(\omega, v, v_*)\right) = v'_*(\omega, v, v_*) + \left(v'(\omega, v, v_*) - v'_*(\omega, v, v_*)\right) \cdot \omega\ \omega \quad (12)$$

$$= v_* + (v - v_*) \cdot \omega\ \omega + (v - v_* - 2(v - v_*) \cdot \omega\ \omega) \cdot \omega\ \omega = v_*. \quad (13)$$

Given that the change of variable associated to the transformation $v' = v'(\omega, v, v_*)$, $v'_* = v'_*(\omega, v, v_*)$ is of the form $dv'dv'_* = dvdv_*$

As a consequence we can rewrite $Q(f, f)$ as

$$Q(f, f) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |(v - v_*) \cdot \omega| \left(f' f'_* - f f_*\right) d\omega\ dv_* \quad (14)$$

We now want to perform the change of variables $\sigma \to \omega$. Indeed, imposing that $v'(\omega, v, v_*) = v''(\sigma, v, v_*)$, we can easily deduce that

$$\sigma = -\frac{v - v_*}{|v - v_*|} + 2\frac{v - v_*}{|v - v_*|} \cdot \omega \quad (15)$$

Recall now that for a generic function $f \in L^1(\mathbb{S}^2)$, the integral is defined in such a way that for any bijective map $\xi \in C^1(U; \mathbb{S}^2)$ with $U \subseteq \mathbb{R}^2$ (a parametrization), we get

$$\int_{\mathbb{S}^2} f(\sigma)\, d\sigma = \int_U f(\xi(x))\, J_\xi(x)\, dx, \quad (16)$$

with

$$J_\xi(x) := \sqrt{\det\left[\nabla_x \xi(x) \left(\nabla_x \xi(x)\right)^T\right]} \quad (17)$$

We consider now the parametrization given by the composition of the change of variables $\omega \to \sigma$ and the parametrization give in polar coordinates. In other words, with a little abuse of notation we define

$$\begin{array}{rcl} \sigma : \left\{\omega \in \mathbb{S}^2 \middle|\ \omega_3 > 0\right\} & \longrightarrow & \mathbb{S}^2 \\ \omega & \longmapsto & -V + 2V \cdot \omega\ \omega, \\ \omega : [-\pi, \pi] \times \left[0, \frac{\pi}{2}\right] & \longrightarrow & \left\{\omega \in \mathbb{S}^2 \middle|\ \omega_3 > 0\right\} \\ (\varphi, \theta) & \longmapsto & \left(\cos\theta \cos(\varphi), \cos\theta \sin(\varphi), \sin\theta\right), \end{array} \tag{18}$$

with $V \in \mathbb{S}^2$.

For reason that will become clear later, we introduce the vectors $e_r(\varphi, \theta)$, $e_\theta(\varphi, \theta)$ and $e_\varphi(\varphi)$ defined as

$$e_r(\varphi, \theta) := \begin{pmatrix} \cos\theta \cos(\varphi) \\ \cos\theta \sin(\varphi) \\ \sin\theta \end{pmatrix}, \tag{19}$$

$$e_\theta(\varphi, \theta) := \partial_\theta\ e_r(\varphi, \theta) = \begin{pmatrix} -\sin\theta \cos(\varphi) \\ -\sin\theta \sin(\varphi) \\ \cos\theta \end{pmatrix}, \tag{20}$$

$$e_\varphi(\varphi) := \frac{1}{\cos\theta} \partial_\varphi e_r(\varphi, \theta) = \begin{pmatrix} -\sin(\varphi) \\ \cos(\varphi) \\ 0 \end{pmatrix}. \tag{21}$$

Given that

$$|e_r(\varphi, \theta)| = |e_\theta(\varphi, \theta)| = |e_\varphi(\varphi)| = 1, \tag{22}$$

$$e_r(\varphi, \theta) \cdot e_\theta(\varphi, \theta) = e_r(\varphi, \theta) \cdot e_\varphi(\varphi) = e_\theta(\varphi, \theta) \cdot e_\varphi(\varphi) = 0, \tag{23}$$

the set $\{e_r(\varphi, \theta), e_\theta(\varphi, \theta), e_\varphi(\varphi)\}$ is a basis for $\mathbb{R}^3$. Moreover, by definition we have that $\omega(\varphi, \theta) = e_r(\varphi, \theta)$.

We then calculate the Jacobian $J_{\sigma \circ \omega}$ as

$$J_{\sigma \circ \omega}(\varphi, \theta) = \sqrt{\det\left[\nabla_{\varphi,\theta}(\sigma \circ \omega(\varphi, \theta)) (\nabla_{\varphi,\theta}(\sigma \circ \omega(\varphi, \theta)))^T\right]} \tag{24}$$

$$= \sqrt{\det\left[(\nabla_{\varphi,\theta}\omega)(\varphi, \theta) (\nabla_\omega \sigma)(\omega(\varphi, \theta)) [(\nabla_\omega \sigma)(\omega(\varphi, \theta))]^T [(\nabla_{\varphi,\theta}\omega)(\varphi, \theta)]^T\right]}. \tag{25}$$

It is easy to calculate the matrices $(\nabla_{\varphi,\theta}\omega)(\varphi, \theta)$ and $(\nabla_\omega \sigma)(\omega)$ as

$$(\nabla_{\varphi,\theta}\omega)(\varphi, \theta) = \begin{pmatrix} \cos\theta\ e_\varphi(\varphi)^T \\ e_\varphi(\varphi, \theta)^T \end{pmatrix}, \tag{26}$$

$$(\nabla_\omega \sigma)(\omega) = 2|V\rangle\langle\omega| + 2V \cdot \omega\, \mathrm{id}, \tag{27}$$

where $(|V\rangle\langle\omega|)_{j,k} := V_j \omega_k$ (and in particular $|V\rangle\langle\omega|v = \omega \cdot v\ V$ for any $v \in \mathbb{R}^3$). As a consequence for any $\omega \in \left\{\omega \in \mathbb{S}^2 \middle|\ \omega_3 > 0\right\}$ we get

$$(\nabla_\omega \sigma)(\omega) [(\nabla_\omega \sigma)(\omega)]^T = 4(|V\rangle\langle\omega| + V \cdot \omega\, \mathrm{id})(|\omega\rangle\langle V| + V \cdot \omega\, \mathrm{id}) \tag{28}$$

$$= 4\left[|V\rangle\langle V| + V \cdot \omega |V\rangle\langle\omega| + V \cdot \omega |\omega\rangle\langle V| + (V \cdot \omega)^2\, \mathrm{id}\right]. \tag{29}$$

Define now $T(\varphi,\theta) = (\nabla_\omega \sigma)(\omega(\varphi,\theta))\left[(\nabla_\omega \sigma)(\omega(\varphi,\theta))\right]^T$; as a consequence we get

$$
\begin{aligned}
&(\nabla_{\varphi,\theta}\omega)(\varphi,\theta)\,(\nabla_\omega \sigma)(\omega(\varphi,\theta))\left[(\nabla_\omega \sigma)(\omega(\varphi,\theta))\right]^T\left[(\nabla_{\varphi,\theta}\omega)(\varphi,\theta)\right]^T = &(30)\\
&= \begin{pmatrix} \cos\theta\ e_\varphi(\varphi)^T \\ e_\varphi(\varphi,\theta)^T \end{pmatrix} T(\varphi,\theta)\left(\cos\theta\ e_\varphi(\varphi), e_\varphi(\varphi,\theta)\right) &(31)\\
&= \begin{pmatrix} (\cos\theta)^2 e_\varphi(\varphi)\cdot T(\varphi,\theta)\, e_\varphi(\varphi) & \cos\theta\ e_\varphi(\varphi)\cdot T(\varphi,\theta)\, e_\theta(\varphi,\theta) \\ \cos\theta\ e_\theta(\varphi,\theta)\cdot T(\varphi,\theta)\, e_\varphi(\varphi) & e_\theta(\varphi,\theta)\cdot T(\varphi,\theta)\, e_\theta(\varphi,\theta) \end{pmatrix}. &(32)
\end{aligned}
$$

Given that $\omega(\varphi,\theta) = e_r(\varphi,\theta)$ and from the fact that $\{e_r(\varphi,\theta), e_\theta(\varphi,\theta), e_\varphi(\varphi)\}$ is a basis we have that for any $v, w \in \operatorname{span}_{\mathbb{R}}\{e_\theta(\varphi,\theta), e_\varphi(\varphi)\}$

$$
v\cdot |V\rangle\langle \omega(\varphi,\theta)|w = w\cdot |V\rangle\langle\omega(\varphi,\theta)|v = 0. \tag{33}
$$

We can finally get

$$
\begin{aligned}
e_\varphi(\varphi)\cdot T(\varphi,\theta)\, e_\varphi(\varphi) &= 4\left[(V\cdot e_\varphi(\varphi))^2 + (V\cdot e_r(\varphi,\theta))^2\right], &(34)\\
e_\varphi(\varphi)\cdot T(\varphi,\theta)\, e_\theta(\varphi,\theta) &= 4\,(V\cdot e_\varphi(\varphi))\,(V\cdot e_\theta(\varphi,\theta)), &(35)\\
e_\theta(\varphi,\theta)\cdot T(\varphi,\theta)\, e_\varphi(\varphi) &= 4\,(V\cdot e_\varphi(\varphi))\,(V\cdot e_\theta(\varphi,\theta)), &(36)\\
e_\theta(\varphi,\theta)\cdot T(\varphi,\theta)\, e_\theta(\varphi,\theta) &= 4\left[(V\cdot e_\theta(\varphi,\theta))^2 + (V\cdot e_r(\varphi,\theta))^2\right]. &(37)
\end{aligned}
$$

From the fact that

$$
\begin{aligned}
&(e_\varphi(\varphi)\cdot T(\varphi,\theta)\, e_\varphi(\varphi))\,(e_\theta(\varphi,\theta)\cdot T(\varphi,\theta)\, e_\theta(\varphi,\theta)) = &(38)\\
&\quad = 16\left[(V\cdot e_\varphi(\varphi))^2 (V\cdot e_\theta(\varphi,\theta))^2 + (V\cdot e_\varphi(\varphi))^2 (V\cdot e_r(\varphi,\theta))^2\right. &(39)\\
&\qquad \left. + (V\cdot e_\theta(\varphi,\theta))^2 (V\cdot e_r(\varphi,\theta))^2 + (V\cdot e_r(\varphi,\theta))^4\right] &(40)\\
&\quad = 16\left[(V\cdot e_\varphi(\varphi))^2 (V\cdot e_\theta(\varphi,\theta))^2 + |V|^2 (V\cdot e_r(\varphi,\theta))^2\right] &(41)\\
&\quad = 16\left[(V\cdot e_\varphi(\varphi))^2 (V\cdot e_\theta(\varphi,\theta))^2 + (V\cdot e_r(\varphi,\theta))^2\right], &(42)\\
&(e_\varphi(\varphi)\cdot T(\varphi,\theta)\, e_\theta(\varphi,\theta))\,(e_\theta(\varphi,\theta)\cdot T(\varphi,\theta)\, e_\varphi(\varphi)) = &(43)\\
&\quad = 16\,(V\cdot e_\varphi(\varphi))^2 (V\cdot e_\theta(\varphi,\theta))^2, &(44)
\end{aligned}
$$

we can explicitly calculate the Jacobian as

$$
\begin{aligned}
J_{\sigma\circ\omega}(\varphi,\theta) &= \sqrt{16\,(\cos\theta)^2 (V\cdot e_r(\varphi,\theta))^2} = 4\cos\theta\ |V\cdot e_r(\varphi,\theta)| &(45)\\
&= 4\cos\theta\ |V\cdot\omega(\varphi,\theta)|. &(46)
\end{aligned}
$$

It is now an easy computation to notice that $J_\omega(\varphi,\theta) = \cos\theta$, and therefore get

$$
\begin{aligned}
\int_{\mathbb{S}^2} f(\sigma)\, d\sigma &= \int_{-\pi}^{\pi}\int_0^{\frac{\pi}{2}} f(\sigma(\omega(\varphi,\theta)))\, J_{\sigma\circ\omega}(\varphi,\omega)\, d\theta\ d\varphi &(47)\\
&= \int_{-\pi}^{\pi}\int_0^{\frac{\pi}{2}} 4\,|V\cdot\omega(\varphi,\theta)|\, f(\sigma(\omega(\varphi,\theta)))\, J_\omega(\varphi,\omega)\, d\theta\ d\varphi &(48)\\
&= \int_{\{\omega\in\mathbb{S}^2 |\ \omega_3>0\}} 4\,|V\cdot\omega|\, f(\sigma(\omega))\, d\omega. &(49)
\end{aligned}
$$

We now apply this to our problem. Using that $(v'', v''_*)(\sigma(\omega)) = (v', v'_*)(\omega)$ we can apply the formula above to get

$$\int_{\mathbb{S}^2} |(v - v_*)\cdot\omega| \left(f' f'_* - f f_*\right) d\omega = |v - v_*| \int_{\mathbb{S}^2} \left| \frac{v - v_*}{|v - v_*|}\cdot\omega \right| \left(f' f'_* - f f_*\right) d\omega \tag{50}$$

$$= 2\,|v - v_*| \int_{\{\omega\in\mathbb{S}^2 |\ \omega_3 > 0\}} \left| \frac{v - v_*}{|v - v_*|}\cdot\omega \right| \left(f' f'_* - f f_*\right) d\omega \tag{51}$$

$$= \frac{|v - v_*|}{2} \int_{\mathbb{S}^2} \left(f'' f''_* - f f_*\right) d\sigma, \tag{52}$$

which implies the exercise.

□

Exercise 3

In this exercise we will study the explicit kernel of a power law potential.

In order to do so, we first introduce some basic properties of motion of a particle in $\mathbb{R}^3$. Let $U : \mathbb{R}_+ \to [0, +\infty)$ a radial potential, and the force $F : \mathbb{R}^3 \to \mathbb{R}$ associated to it defined as

$$F(x) := -\nabla_x \left(U(|x|)\right). \tag{53}$$

A particle submitted to F satisfies Newton's equation, in the sense that its position and velocity $(x(t), v(t))$ solve

$$\begin{cases} \partial_t x(t) = v(t), \\ \partial_t v(t) = F(x(t)). \end{cases} \tag{54}$$

Once fixed the initial condition $(x(0), v(0)) = (x_0, v_0)$ we know the solution to (54) is unique.

(i) Prove that the *angular momentum*[1] $L(t) := x(t) \wedge v(t)$ is conserved. Prove that the movement of the particle lies in a plane.

Hint: For two generic vectors $u,\ w \in \mathbb{R}^3$ what geometical property do $u,\ w$ and $u \wedge w$ fulfill?

(ii) Let $\mathcal{E}_c(t)$ and $\mathcal{E}_p(t)$ be respectively the *kinetic and potential energy of the particle* at time t, i.e.

$$\mathcal{E}_c(t) = \frac{1}{2}|v(t)|^2, \qquad \mathcal{E}_p(t) = U(|x(t)|). \tag{56}$$

Show that the total energy of the system $\mathcal{E}_{tot}(t) = \mathcal{E}_c(t) + \mathcal{E}_p(t)$ is conserved in time if $(x(t), v(t))$ is a solution of (54).

[1] Recall that given two vectors $u, w \in \mathbb{R}^3$ with $u \wedge w$ we denote the **vector product** between u and w, which is defined as

$$u \wedge w = \begin{pmatrix} u_2 w_3 - u_3 w_2 \\ u_3 w_1 - u_1 w_3 \\ u_1 w_2 - u_2 w_1 \end{pmatrix}. \tag{55}$$

(iii) From point (i) the motion of the particle lays in the plain spanned by x_0 and v_0. Consider the system of coordinates so that the component along the third component is zero. Furthermore on the plain of motion consider polar coordinates, so that any vector x can be represented as $x = (\rho\cos\alpha, \rho\sin\alpha, 0)$ in a suitable basis. Let $\rho(t),\ \alpha(t)$ the polar coordinates associated to $x(t)$ (i.e. $x(t) = (\rho(t)\cos\alpha(t), \rho(t)\sin\alpha(t), 0)$). Find the expression of $\mathcal{E}_c(t)$ and $\mathcal{E}_{tot}(t)$ in terms of $\rho(t),\ \alpha(t)$.

Assume now that U is compactly supported, that is $U(\rho) = 0$ for $\rho > \sigma$ for some real $\sigma > 0$ and decreasing in ρ. Let us assume in addition that $|x_0| > \sigma$, $v_0 = -Ve_1$ with $V > 0$.

For small times the motion of the particle is free (as long as we are outside of the support of the potential $v(t)$ is constant); we assume that initially the particle approaches the origin with impact parameter $p \in (0, \sigma)$, where the impact parameter is defined as $p = x_0 \cdot e_2$ (i.e., the trajectory can be written for small times as $x(t) = (t - C)v_0 + pe_2$ with a suitable real constant C, see also Figure 1 below).

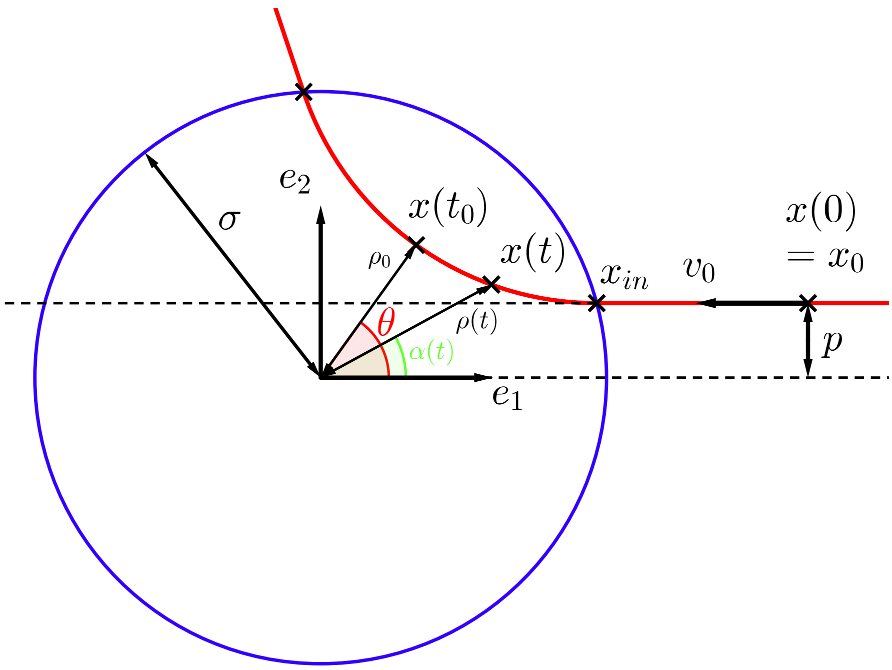

Figure 1: The movement of the particle through the support of the potential.

(iv) Using the repulsive property of the potential, prove that the distance ρ between the particle and the origin has a single minimum ρ_0.

Suppose that t_0 denotes the time at which the minimum is reached. Consider the line between the origin and $x(t_0)$ (the so-called *apse line*), and define as θ the angle between e_1 and this line. The angle θ is called the *deviation angle*.

(v) Prove that the trajectory of $x(t)$ is symmetric with respect to this minimum, i.e. we have for any $t \in \mathbb{R}$

$$\rho(t_0+t) = \rho(t_0-t), \qquad \alpha(t_0+t) - \theta = -(\alpha(t_0-t) - \theta). \tag{57}$$

(vi) In the case of the potential with cut-off, prove that the conservation of the total energy and the angular momentum respectively write:

$$\begin{cases} \frac{1}{2}\left(\dot{\rho}^2 + \rho^2\dot{\alpha}^2\right) + U(\rho) = \frac{1}{2}V^2 + U(\sigma), \\ \rho^2\dot{\alpha} = pV, \end{cases} \tag{58}$$

where $\dot{\rho}$ and $\dot{\alpha}$ denote respectively the time derivatives of ρ and α.

Hint: Consider the total energy and the angular momentum at the point x_{in}, where the particle enters the support of the potential (that is, the first time that $|x(t)| = \sigma$).

(vii) We denote as t_1 the time such that $x(t_1) = x_{in} = \sigma(\cos\alpha(t_1), \sin\alpha(t_1), 0)$. Prove that

$$\theta = \int_{t_1}^{t_0} \dot{\alpha}(t)\,dt + \arcsin\left(\frac{p}{\sigma}\right). \tag{59}$$

(viii) Prove the following identity:

$$\int_{t_1}^{t_0} \dot{\alpha}(t)\,dt = \frac{pV}{\sqrt{2}} \int_{\rho_0}^{\sigma} \frac{1}{w^2\sqrt{\frac{V^2}{2}\left(1-\frac{p^2}{w^2}\right) - U(w) + U(\sigma)}}dw. \tag{60}$$

Hint: Use the conservation laws (58) to find an expression for $\dot{\rho}$ and $\dot{\alpha}$ in terms of ρ only, write $\dot{\alpha} = \frac{\dot{\alpha}}{\dot{\rho}}\dot{\rho}$, substitute $\frac{\dot{\alpha}}{\dot{\rho}}$ with a function of ρ only, integrate in time and change variables as $\rho(t) = w$.

(ix) Find an equation satisfied by the minimal distance ρ_0. Up to assume that we can solve this equation, deduce an explicit expression of θ (the expression (60) is of course not explicit, since it relies on determining the quantity $\dot{\alpha}$).

Consider now $U(\rho) = k\rho^{1-n}$ in its support. The explicit expression of θ reads:

$$\theta = \frac{pV}{\sqrt{2}} \int_{\rho_0}^{\sigma} \frac{1}{w^2\sqrt{\frac{V^2}{2}\left(1-\frac{p^2}{w^2}\right) - \frac{k}{w^{n-1}} + \frac{k}{\sigma^{n-1}}}}dw + \arcsin\left(\frac{p}{\sigma}\right). \tag{61}$$

(x) Prove that, thanks to a change of variables, the deviation angle θ can be written as:

$$\theta = \int_{\lambda}^{\bar{x}} \frac{1}{\sqrt{1-x^2-\left(\frac{x}{b}\right)^{n-1}}}dx + \arcsin\left(\frac{p}{\sigma}\right), \tag{62}$$

with

$$\lambda = \frac{p}{\sigma}\sqrt{1+\frac{2k}{V^2\sigma^{n-1}}}, \qquad b = p\left(\frac{V^2}{2k} + \frac{k}{\sigma^{n-1}}\right)^{\frac{1}{n-1}}, \tag{63}$$

and $\bar{x}$ solving the equation $1 - \bar{x}^2 - \left(\frac{\bar{x}}{b}\right)^{n-1} = 0$.

(xi) Finally consider the limit $\sigma \to +\infty$ (which corresponds to relaxing the cut-off on the support of the potential). Recall that the collision kernel is written as

$$B(\theta, V) = Vp(\theta)\,\partial_\theta p(\theta). \tag{64}$$

Prove that in the case of the inverse power law potential $U(\rho) = k\rho^{1-n}$ **without cut-off**, the collision kernel has the form:

$$B(\theta, V) = V^\gamma b(\theta), \tag{65}$$

with $\gamma = \frac{n-5}{n-1}$, and where b is seen as a function of θ through (62).

Proof. We start from (i). Given that the force can be explicitly written as $F(x) = -U'(|x|)\frac{x}{|x|}$, it is parallel to the vector $x(t)$ for every time. Therefore from general properties of the vector product we get

$$\partial_t L(t) = (\partial_t x(t)) \wedge v(t) + x(t) \wedge (\partial_t v(t)) = v(t) \wedge v(t) + x(t) \wedge F(x(t)) = 0, \tag{66}$$

and the angular momentum is conserved. Now, in general we have that $a \wedge b$ is both orthogonal to a and b, so if $L(t)$ is constant, this means that the plain on which the dynamics happens is fixed as the orthogonal plane to $L(0)$.

To prove (ii) we differentiate $\mathcal{E}_{tot}(t)$ to get

$$\partial_t \mathcal{E}_{tot}(t) = v(t) \cdot \partial_t v(t) + F(x(t)) \cdot v(t) = 0 \tag{67}$$

To prove (iii), in analogy to the previous exercise we define the following vectors:

$$e_r(\alpha) := \begin{pmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{pmatrix}, \qquad e_\alpha(\alpha) := \begin{pmatrix} -\sin\alpha \\ \cos\alpha \\ 0 \end{pmatrix}. \tag{68}$$

Clearly $e_r \cdot e_\alpha = 0$, $\partial_\alpha e_r = e_\alpha$ and $\partial_\alpha e_\alpha = -e_r$. Furthermore we have $x(t) = \rho(t)\, e_r(\alpha(t))$, and therefore, given that $v(t) = \partial_t x(t)$, the polar decomposition of $v(t)$ is given as

$$v(t) = \dot\rho(t)\, e_r(\alpha(t)) + \rho(t)\,\dot\alpha(t)\, e_\alpha(\alpha(t)). \tag{69}$$

From this it is easy to see that $|v(t)|^2 = |\dot\rho(t)|^2 + |\rho(t)|^2 |\dot\alpha(t)|^2$. As a consequence the energy can be written as

$$\mathcal{E}_{tot}(t) = \frac{1}{2}|\dot\rho(t)|^2 + \frac{1}{2}|\rho(t)|^2 |\dot\alpha(t)|^2 + U(\rho(t)) \tag{70}$$

To solve (iv) we can now write the derivative of v as

$$\dot v(t) = \ddot\rho(t)\, e_r(\alpha(t)) + 2\dot\rho(t)\,\dot\alpha(t)\, e_\alpha(\alpha(t)) \tag{71}$$

$$+ \rho(t)\,\ddot\alpha(t)\, e_\alpha(\alpha(t)) - \rho(t)\,(\dot\alpha(t))^2\, e_r(\alpha(t)). \tag{72}$$

Now, we can also write the force term as

$$F(x(t)) = -U'(\rho(t))\, e_r(\alpha(t)), \tag{73}$$

which means we can rewrite (54) as

$$\begin{cases} \ddot{\rho} - \rho\dot{\alpha}^2 = -U'(\rho), \\ 2\dot{\rho}\dot{\alpha} + \rho\ddot{\alpha} = 0. \end{cases} \tag{74}$$

Given that the second equation can be written as $\partial_t \left(\rho^2 \dot{\alpha}\right) = 0$, this means that $\dot{\alpha}(t) = \frac{C}{\rho^2}$ for a suitable constant C. We substitute this information in the first term in (74) to get

$$\begin{cases} \ddot{\rho} = \frac{C^2}{\rho^3} - U'(\rho), \\ \dot{\alpha} = \frac{C}{\rho^2}. \end{cases} \tag{75}$$

Notice now that if we consider the second derivative of the second component of x we get

$$\ddot{x}_2(t) = \left(\ddot{\rho}(t) - \rho(t)(\dot{\alpha}(t))^2\right)\sin\alpha(t) + (2\dot{\rho}(t)\dot{\alpha}(t) + \rho(t)\ddot{\alpha}(t))\cos\alpha(t) \tag{76}$$

$$= \left(\ddot{\rho}(t) - \rho(t)(\dot{\alpha}(t))^2\right)\sin\alpha(t) = -U'(\rho(t))\sin\alpha(t) \tag{77}$$

$$= \left|U'(\rho(t))\right|\sin\alpha(t), \tag{78}$$

where in the last equality we used the fact that U is decreasing.

We now show that if $x_2(0) > 0$ and $\dot{x}_2(0) = 0$ (which is our case), then ρ can never vanish. To do so we will show something stronger, that is that x_2 can only increase. Assume for example that there exists a time $\tau_0 > 0$ such that $x_2(\tau_0) < x_2(0)$. By continuity of x_2 and up to choosing a different (smaller) value of τ_0, we can assume that $x_2(s) > 0$ for any $s \in [0, \tau_0]$. The Rolle's theorem implies now that there exists a time $\tau_1 \in (0, \tau_0)$ such that $\dot{x}_2(\tau_1) < 0$. Given that $\dot{x}_2(0) = 0$ again by assumption, we get that applying Rolle's theorem once more, there exists a value $\tau_2 \in (0, \tau_1)$ such that $\ddot{x}_2(\tau_2) < 0$. From (76) we get that $\sin\alpha(\tau_2) < 0$ which implies that $x_2(\tau_2) < 0$, which is a contradiction. Therefore $x_2(\tau) \geqslant x_2(0)$ for any $\tau \geqslant 0$.

This last fact with (75) implies that $\ddot{\rho} > 0$ always. This implies that if a zero for $\dot{\rho}$ (and therefore a minimum for ρ) exists, it must be unique. We now show that ρ has a minimum. Indeed, suppose that there exists a radius R such that for any $t > 0$ we have $\rho(t) \leqslant R$. We then get

$$\rho(t) = \rho(0) + t\dot{\rho}(0) + \int_0^t (t-s)\,\ddot{\rho}(s)\,ds \geqslant \rho(0) + t\dot{\rho}(0) + \int_0^t (t-s)\,\frac{C}{R^3}ds \tag{79}$$

$$= \rho(0) + t\dot{\rho}(0) + \frac{Ct^2}{2R^3}. \tag{80}$$

This implies that for t large enough we have $\rho > R$, which is a contradiction and proves that there exists a time $\tau > 0$ such that $\rho(\tau) > \rho(0)$. Given that initially $\dot{\rho}(0) < 0$, this implies that there exists a minimum point for ρ.

Now, as hinted in the text, let's denote with t_0 the time such that $\dot{\rho}(t_0) = 0$. To solve (v) now, notice that both $\rho(t_0 + t)$ and $\rho(t_0 - t)$ solve the problem

$$\begin{cases} \ddot{\gamma} = \frac{C^2}{\gamma^3} - U'(\gamma), \\ \gamma(0) = \rho(t_0), \\ \dot{\gamma}(0) = 0, \end{cases} \tag{81}$$

therefore $\rho(t_0+t)=\rho(t_0-t)$.

Now, define as in the text $\theta=\alpha(t_0)$; we have that $\dot{\alpha}=\frac{C}{\rho^2}$, so we get

$$\alpha(t_0+t)-\theta=\int_0^t \partial_s\alpha(t_0+s)\,ds=\int_0^t \frac{C}{(\rho(t_0+s))^2}ds=\int_0^t \frac{C}{(\rho(t_0-s))^2}ds \tag{82}$$

$$=-\int_0^t \partial_s\alpha(t_0-t)\,ds=-(\alpha(t_0-t)-\theta). \tag{83}$$

To solve (vi) we get that

$$x(t)\wedge v(t)=\begin{pmatrix}0\\0\\\rho(t)^2\dot{\alpha}(t)\end{pmatrix}. \tag{84}$$

Moreover at initial time we have that the angular momentum is given as

$$x_0\wedge v_0=((C-t)v_0+pe_2)\wedge(-Ve_1)=pV. \tag{85}$$

Together with conservation of the energy we obtain (58).

To solve (vii) it is enough to observe that by definition $\theta=\alpha(t_0)$ and that $\alpha(t_1)=\arcsin\left(\frac{x(t_1)\cdot e_2}{|x(t_1)|}\right)=\arcsin\left(\frac{p}{\sigma}\right)$, and therefore

$$\theta=\alpha(t_1)=\int_{t_1}^{t_0}\dot{\alpha}(t)\,dt+\alpha(t_1)=\int_{t_1}^{t_0}\dot{\alpha}(t)\,dt+\arcsin\left(\frac{p}{\sigma}\right). \tag{86}$$

To prove (viii) we get that from the conservation of momentum we get $\dot{\alpha}=\frac{pV}{\rho^2}$. Substituting this in the equation for the conservation of the energy we get

$$|\dot{\rho}|=\sqrt{2\left[\frac{1}{2}V^2+U(\sigma)-\frac{p^2V^2}{2\rho^2}-U(\rho)\right]} \tag{87}$$

$$=\sqrt{2}\sqrt{\frac{V^2}{2}\left(1-\frac{p^2}{\rho^2}\right)-U(\rho)+U(\sigma)}. \tag{88}$$

As a consequence we now get

$$\int_{t_1}^{t_0}\dot{\alpha}(t)\,dt=\int_{t_1}^{t_0}\frac{pV}{(\rho(t))^2}dt \tag{89}$$

$$=\int_{t_1}^{t_0}\frac{pV}{(\rho(t))^2}\frac{1}{\sqrt{2}\sqrt{\frac{V^2}{2}\left(1-\frac{p^2}{(\rho(t))^2}\right)-U(\rho(t))+U(\sigma)}}|\dot{\rho}(t)|\,dt \tag{90}$$

$$=\frac{pV}{\sqrt{2}}\int_{\rho_0}^{\sigma}\frac{1}{w^2\sqrt{\frac{V^2}{2}\left(1-\frac{p^2}{w^2}\right)-U(w)+U(\sigma)}}dw, \tag{91}$$

where in the last equality we used the fact that between t_1 and t_0 we have $|\rho(t))=-\rho(t)$ and used the change of variables $\rho(t)=w$.

To solve (ix) we notice that $\dot{\rho}(t_0) = 0$ and 58 to get that ρ_0 must solve

$$\frac{p^2V^2}{2\rho_0^2} + U(\rho_0) = \frac{V^2}{2} + U(\sigma), \tag{92}$$

and therefore

$$\frac{V^2}{2}\left(1 - \frac{p^2}{\rho_0^2}\right) = U(\sigma) - U(\rho_0). \tag{93}$$

The formula for θ is now given as

$$\theta = \frac{pV}{\sqrt{2}} \int_{\rho_0}^{\sigma} \frac{1}{w^2\sqrt{\frac{V^2}{2}\left(1 - \frac{p^2}{w^2}\right) - U(w) + U(\sigma)}} dw + \arcsin\left(\frac{p}{\sigma}\right). \tag{94}$$

To prove (x) we first look for C, γ, b such that if $x = \frac{\gamma}{w}$ we have

$$\frac{V^2}{2}\left(1 - \frac{p^2}{w^2}\right) - \frac{k}{w^{n-1}} + \frac{k}{\sigma^{n-1}} = C\left(1 - x^2 - \left(\frac{x}{b}\right)^{n-1}\right). \tag{95}$$

This implies

$$\begin{cases} C = \frac{V^2}{2} + \frac{k}{\sigma^{n-1}}, \\ C\gamma^2 = \frac{p^2V^2}{2}, \\ C\left(\frac{\gamma}{b}\right)^{n-1} = k. \end{cases} \tag{96}$$

The solution of this system is given by $C = \frac{V^2}{2} + \frac{k}{\sigma^{n-1}}$, $\gamma = p\left(1 + \frac{2k}{V^2\sigma^{n-1}}\right)^{-\frac{1}{2}}$ and $b = \frac{pV}{\sqrt{2}k^{\frac{1}{n-1}}}\left(\frac{V^2}{2} + \frac{k}{\sigma^{n-1}}\right)^{-\frac{n-3}{2(n-1)}}$.

We now change variables in the integral for θ to get

$$\frac{pV}{\sqrt{2}} \int_{\rho_0}^{\sigma} \frac{1}{w^2\sqrt{\frac{V^2}{2}\left(1 - \frac{p^2}{w^2}\right) - \frac{k}{w^{n-1}} + \frac{k}{\sigma^{n-1}}}} dw = \tag{97}$$

$$= \int_{\frac{\gamma}{\sigma}}^{\frac{\gamma}{\rho_0}} \frac{1}{\sqrt{1 - x^2 - \left(\frac{x}{b}\right)^{n-1}}} dx. \tag{98}$$

If $\overline{x} = \frac{\gamma}{\rho_0}$ the equation for $\overline{x}$ comes from the equation for ρ_0.

Finally, to prove (xi) we get that, after performing the limit $\sigma \to \infty$, b and θ solve

$$\theta = \int_0^{\overline{x}} \frac{1}{\sqrt{1 - x^2 - \left(\frac{x}{b}\right)^{n-1}}} dx \tag{99}$$

where now $\overline{x}$ solves

$$1 - \overline{x}^2 - \left(\frac{\overline{x}}{\overline{b}}\right)^{n-1} = 0, \tag{100}$$

and where $\overline{b} = p\left(\frac{V^2}{2k}\right)^{\frac{1}{n-1}}$; as a consequence, p as function of θ can be seen as $p(\theta) = \left(\frac{V^2}{2k}\right)^{-\frac{1}{n-1}} \overline{b}(\theta)$. This allows us to conclude that

$$B(\theta, V) = Vp(\theta)\,\partial_\theta p(\theta) = V\left(\frac{V^2}{2k}\right)^{-\frac{2}{n-1}} \overline{b}(\theta)\,\partial_\theta \overline{b}(\theta) = V^{\frac{n-5}{n-1}} B(\theta), \tag{101}$$

with a suitable $B(\theta)$.

□